Research Methodology

UNIT 3

Gamma distribution

Defn:

Let x be a random variable having the p.d.f $f(x) = \frac{1}{\Gamma\alpha \; \beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$, $0 < x < \infty$. Then x is said to have a Gamma distribution with parameter α and β and it denoted by $\Gamma(\alpha, \beta)$

Note:

$f(x)$ is a p.d.f

Proof:

i) Clearly $f(x) \geq 0$.

ii) To Prove: $\int_0^{\infty} f(x)\,dx = 1$

From calculus we have seen that $\Gamma\alpha = \int_0^{\infty} e^{-y} y^{\alpha-1}\,dy$

Put $\alpha = 1$

$$\Gamma 1 = \int_0^{\infty} e^{-y}\,dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty} = -e^{-\infty} + e^{0} = 1$$

If $\alpha > 1$, then by integration by parts shows that

$$\Gamma\alpha = \int_0^{\infty} e^{-y} (\alpha-1) y^{\alpha-2}\,dy$$

$$= (\alpha-1) \int_0^{\infty} e^{-y} y^{\alpha-2}\,dy$$

$$\Gamma\alpha = (\alpha-1)\Gamma(\alpha-1)$$

Accordingly if α is a +ve integer > 1

Then $\Gamma\alpha = (\alpha-1)!$

Let $y = \frac{x}{\beta}$ where $\beta > 0$

$dy = \frac{dx}{\beta}$

$$\Gamma\alpha = \int_0^\infty e^{-x/\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} \frac{dx}{\beta}$$

$$\Gamma\alpha = \int_0^\infty e^{-x/\beta} \frac{x^{\alpha-1}}{\beta^\alpha}\, dx$$

$$1 = \int_0^\infty \frac{1}{\Gamma\alpha \cdot \beta^\alpha} x^{\alpha-1} e^{-x/\beta} \cdot dx$$

$$\int_0^\infty f(x)\, dx = 1$$

$\therefore f(x)$ is a p.d.f.

Exponential distribution.

In the Gamma distribution, let $\alpha = 1$, then

$$f(x) = \frac{1}{\Gamma\alpha\, \beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty$$

$$f(x) = \frac{1}{\Gamma 1\;\; \beta} x^{1-1} e^{-x/\beta}$$

$$f(x) = \frac{1}{\beta} e^{-x/\beta}, \quad 0 < x < \infty.$$

It is called a exponential distribution

Derive Moment generating function of Gamma distribution

Let x be a random variable

$$M(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{1}{\Gamma\alpha\, \beta^\alpha} x^{\alpha-1} e^{-x/\beta} \cdot dx$$

$$= \int_0^\infty \frac{1}{\Gamma\alpha\, \beta^\alpha} x^{\alpha-1} e^{-x\left(\frac{1}{\beta} - t\right)} dx$$

$$M(t) = \int_0^\infty \frac{1}{\Gamma\alpha \, \beta^\alpha} \cdot x^{\alpha-1} \, e^{\frac{-x}{\beta}(1-\beta t)} \, dx$$

Put $y = \dfrac{x(1-\beta t)}{\beta}$

$$x = \frac{y\beta}{1-\beta t} \Rightarrow dx = \frac{\beta \, dy}{1-\beta t}$$

$$M(t) = \int_0^\infty \frac{1}{\Gamma\alpha \, \beta^\alpha} \left(\frac{y\beta}{1-\beta t}\right)^{\alpha-1} \frac{\beta}{1-\beta t} e^{-y} \, dy$$

$$= \int_0^\infty \frac{1}{\beta^\alpha \, \Gamma\alpha} \frac{y^{\alpha-1} \cdot \beta^{\alpha-1} \cdot \beta}{(1-\beta t)^{\alpha-1}(1-\beta t)} e^{-y} \, dy$$

$$= \int_0^\infty \frac{1}{\Gamma\alpha} \frac{y^{\alpha-1} \cdot e^{-y}}{(1-\beta t)^\alpha} \, dy$$

$$= \frac{1}{\Gamma\alpha \, (1-\beta t)^\alpha} \int_0^\infty e^{-y} \cdot y^{\alpha-1} \cdot dy$$

$$M(t) = \frac{1}{\Gamma\alpha \, (1-\beta t)^\alpha} \Gamma\alpha = \frac{1}{(1-\beta t)^\alpha}$$

$$M(t) = (1-\beta t)^{-\alpha}$$

Mean and variance of Gamma distribution

$$M(t) = (1-\beta t)^{-\alpha}$$

$$M'(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta)$$

$$M'(t) = \alpha\beta(1-\beta t)^{-\alpha-1}$$

Put $t=0$, $M'(0) = \alpha\beta$

i.e.) $\boxed{\mu = \alpha\beta}$

$$M''(t) = (-\alpha-1)\alpha\beta(1-\beta t)^{-\alpha-2}(-\beta)$$

$$MM''(t) = \alpha\beta^2(\alpha+1)(1-\beta t)^{-\alpha-2}$$

$$MM''(0) = \alpha\beta^2(\alpha+1)$$

$$= \alpha^2\beta^2 + \alpha\beta^2$$

$$\sigma^2 = \mu''(0) - (\mu'(0))^2$$

$$\sigma^2 = \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

$$\Rightarrow \boxed{\sigma^2 = \alpha\beta^2}$$

Example 1

Let x be a random variable such that $E(x^m) = \frac{(m+3)!\,3^m}{3!}$, where $m = 1, 2, \ldots$ Find moment generating function of x.

Soln:

$$\text{W.K.T}\quad E(x^m) = \frac{(m+3)!\,3^m}{3!}$$

$E(x^r)$ is the coefficients of $\frac{t^r}{r!}$ in $\mu(t)$.

$$M(t) = E(x^0)\frac{t^0}{0!} + E(x^1)\frac{t^1}{1!} + E(x^2)\frac{t^2}{2!} + \cdots$$

$$= 1 + \frac{4!}{3!}\frac{(3t)}{1!} + \frac{5!}{3!}\frac{(3t)^2}{2!} + \frac{6!}{3!}\frac{(3t)^3}{3!} + \cdots$$

$$= 1 + \binom{4}{1}3t + \binom{5}{2}(3t)^2 + \binom{6}{3}(3t^3) + \cdots$$

$M(t) = (1-3t)^{-4}$. Which is the moment generating function of the Gamma distribution with $\alpha = 4$ and $\beta = 3$

Waiting time Distribution:

Let the random variable W be the time that is needed to obtain exactly k changes

Then W follows a Gamma distribution

The distribution function of w is given by

$G(w) = P_r(W \leq w)$

$= 1 - P_r(W > w)$

Now, the event $W > w$, $w > 0$ is equivalently to the event that there are less than k-changes in the interval of length w

(a) If the random variable x is the number of changes in an interval of length w, then $P_r(W > w) = \sum_{x=0}^{k-1} P_r(X = x)$

$$= \sum_{x=0}^{k-1} g(x, w)$$

$$= \sum_{x=0}^{k-1} \frac{e^{-\lambda w}(\lambda w)^x}{x!} \quad \text{--- (1)}$$

$$\text{W.K.T., } \int_{\mu}^{\infty} \frac{1}{\Gamma k} e^{-z} z^{k-1} dz = \sum_{x=0}^{k-1} \frac{e^{-\mu} \mu^x}{x!}$$

Put $\mu = \lambda w$

$$\int_{\lambda w}^{\infty} \frac{1}{\Gamma k} e^{-z} z^{k-1} dz = \sum_{x=0}^{k-1} \frac{e^{-\lambda w}(\lambda w)^x}{x!} \quad \text{--- (2)}$$

From (1) & (2),

$$P_r(W > w) = \int_{\lambda w}^{\infty} \frac{1}{\Gamma k} e^{-z} z^{k-1} dz$$

$$G(w) = 1 - P_r(W > w)$$

$$G(\omega) = 1 - \int_{\lambda\omega}^{\infty} \frac{1}{\Gamma k} \cdot e^{-z} \cdot z^{k-1} dz \quad \text{--- } (*)$$

Now, $\Gamma k = \int_0^{\infty} e^{-z} \cdot z^{k-1} dz$

$$1 = \int_0^{\infty} \frac{1}{\Gamma k} e^{-z} z^{k-1} dz$$

$$1 = \int_0^{\lambda\omega} \frac{1}{\Gamma k} \cdot e^{-z} z^{k-1} dz + \int_{\lambda\omega}^{\infty} \frac{1}{\Gamma k} e^{-z} \cdot z^{k-1} dz$$

$$G(\omega) = 1 - \int_{\lambda}^{\omega} \frac{1}{\Gamma k} e^{-z} z^{k-1} dz$$

Put $z = \lambda y$

$dz = \lambda dy$

$$G(\omega) = \int_0^{\infty} \frac{1}{\Gamma k} e^{-\lambda y} (\lambda y)^{k-1} \lambda dy$$

$$= \int_0^{\infty} \frac{1}{\Gamma k} e^{-\lambda y} \cdot \lambda^{k-1} y^{k-1} \lambda dy$$

$$G(\omega) = \int_0^{\infty} \frac{1}{\Gamma k \left(\frac{1}{\lambda}\right)^k} e^{-y/\frac{1}{\lambda}} y^{k-1} dy$$

$g(\omega) = G'(\omega)$

$$g(\omega) = \begin{cases} \frac{1}{\Gamma k \left(\frac{1}{\lambda}\right)^k} e^{-y/\frac{1}{\lambda}} y^{k-1}, & 0 < \omega < \infty \\ 0 & \text{otherwise} \end{cases}$$

which is a P.d.f of Gamma of distribution with $\alpha = k$ and $\beta = 1/\lambda$. Hence a waiting time distribution followed a Gamma distribution.

Examples:

Let the waiting time W have a Gamma P.d.f with $\alpha = k$ and $\beta = \frac{1}{\lambda}$.

accordingly $E(W) = \frac{k}{\lambda}$ is $k=1$, then $E(W) = \frac{1}{\lambda}$

i) The expected waiting time for $k=1$ changes is equal to the reciprocal of λ.

Chi-Square distribution (χ^2).

In the P.d.f of Gamma distribution take $\alpha = \frac{r}{2}$ and $\beta = 2$.

$$f(x) = \begin{cases} \frac{1}{\Gamma\frac{r}{2}\, 2^{\frac{r}{2}}} x^{\frac{r}{2}-1} \cdot e^{-\frac{x}{2}} & , 0<x<\infty \\ 0 & \text{otherwise} \end{cases}$$

Then $f(x)$ is called the P.d.f of Chi-Square distribution with Parameter r and it is denoted by $\chi^2(r)$

In a Gamma distribution, $M(t) = (1-\beta t)^{-\alpha}$.

we have in a chi-square distribution

$$M(t) = (1-2t)^{-\frac{r}{2}}$$

we have mean of Gamma distribution $\mu = \alpha\beta$ and variation $\sigma^2 = \alpha\beta^2$

$\therefore$ In a Chi-Square distribution mean $\mu = r$ and variation $\sigma^2 = \frac{r}{2} \cdot 4$

$$\mu = r \text{ and } \sigma^2 = 2r.$$

Problem:

If X has the p.d.f (i) $f(x) = \frac{1}{\Gamma 2 \cdot 4} x e^{-x/2}, 0 < x < \infty$

then X is $\chi^2(4)$

ii) Find m.g.f mean and variations

Soln:

Given $f(x) = \frac{1}{\Gamma 2 \cdot 4} x e^{-x/2}, 0 < x < \infty$

Here $\Gamma \alpha = \Gamma 2$

$\therefore \alpha = 2$

In $\chi^2(r)$ $\alpha = \frac{r}{2} \Rightarrow r = 4$

$\therefore x$ is $\chi^2(4)$

$\left[\because M(t) = (1-\beta t)^{-\alpha} \quad \beta = 2, \alpha = \frac{r}{2}\right]$

$M(t) = (1-2t)^{-r/2}$

$\therefore M(t) = (1-2t)^{-4/2}$

$M(t) = (1-2t)^{-2}$

mean $\mu = r$ $\quad [\because \mu = \alpha\beta]$

$\Rightarrow \mu = 4$

$\sigma^2 = 2r = 2 \times 4$

$\sigma^2 = 8$ $\quad [\because \sigma^2 = \alpha\beta^2]$

problem :

If x has a m.g.f $M(t) = (1-2t)^{-8}$, $1 < t < \frac{1}{2}$

Find $\chi^2(r)$

soln:

Given $M(t) = (1-2t)^{-8}$

$\alpha = 8$, $\beta = 2$

$\alpha = \frac{r}{2} \Rightarrow 8 = \frac{r}{2} \Rightarrow r = 16$

X is $\chi^2(16)$

Note :

If the random variable X is $\chi(r)$, then $c_1 < c_2$ we have $P_r(c_1 \le X \le c_2) = P_r(X \le c_2) - P_r(X \le c_1)$

problem:

Let x be $\chi^2(10)$ Find $P_r(3.25 \le X \le 20.5)$

soln:

Given X is $\chi^2(10)$

$P_r(3.25 \le X \le 20.5) = P_r(X \le 20.5) - P_r(X \le 3.25)$

$= 0.975 - 0.025$

$P_r(3.25 \le X \le 20.5) = 0.950$

Problem:

If X is $\chi^2(5)$ Determine the constant c and d so that $P_r(c < X < d) = 0.95$ and $P_r(X < c) = 0.025$

Soln:

Given X is $\chi^2(5)$

$\therefore r = 5$

Given $P_r(X < c) = 0.025$

$\therefore c = 0.831$

Now, $P_r(c < X < d) = 0.95$

i.e) $P_r(X < d) - P_r(X < c) = 0.95$

$P_r(X < d) - (0.025) = 0.95$

$P_r(X < d) = 0.95 + 0.025$

$P_r(X < d) = 0.975$

$P_r(X < d) = 12.8$

$d = 12.8$

Problem:

Let X have a Gamma distribution with $\alpha = \frac{r}{2}$ where r is a +ve integer & $\beta > 0$ define the random variable $Y = \frac{2X}{\beta}$. Find the p.d.f of Y

Soln:

Given X have a Gamma distribution with $\alpha = \frac{r}{2}$ and $\beta > 0$

$\therefore$ p.d.f of X is $f(x) = \frac{1}{\Gamma\alpha \, \beta^{\alpha}} x^{\alpha - 1} e^{-\frac{x}{\beta}}, \; 0 < x < \infty$

i.e) $f(x) = \begin{cases} \frac{1}{\Gamma\frac{r}{2} \, \beta^{r/2}} x^{\frac{r}{2} - 1} e^{-\frac{x}{\beta}}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$

Given $Y = \frac{2X}{\beta}$

$\therefore$ The distribution function of Y is

$G(y) = P_r(Y \le y) = P_r\left(\frac{2X}{\beta} \le y\right)$

$G(y) = P_r\left(X \le \frac{y\beta}{2}\right)$

If $y < 0$, then $G(y) = 0$

If $y > 0$, then $G(y) = P_r\left(X \le \frac{y\beta}{2}\right)$

$$G(y) = \int_0^{y\beta/2} \frac{1}{\Gamma\frac{r}{2}\,\beta^{r/2}}\, x^{\frac{r}{2}-1} e^{-\frac{x}{\beta}}\, dx$$

put $y = \frac{2x}{\beta} \Rightarrow x = \frac{\beta y}{2}$

$dx = \frac{\beta\, dy}{2}$, $x = 0 \Rightarrow y = 0$

$x = \frac{y\beta}{2} \Rightarrow y = \frac{y\beta}{2} \cdot \frac{2}{\beta} = y$

$$G(y) = \int_0^{y} \frac{1}{\Gamma\frac{r}{2}\,\beta^{r/2}} \left(\frac{\beta y}{2}\right)^{\frac{r}{2}-1} e^{-\frac{\beta y}{2\beta}} \cdot \frac{\beta\, dy}{2}$$

$$= \int_0^{y} \frac{1}{\Gamma\frac{r}{2}\,\beta^{r/2}} \frac{\beta^{r/2-1}\, y^{r/2-1}}{2^{r/2-1}} e^{-\frac{y}{2}} \frac{\beta}{2}\, dy$$

$$G(y) = \int_0^{y} \frac{1}{\Gamma\frac{r}{2}} \frac{y^{r/2-1} e^{-\frac{y}{2}}}{2^{r/2}}\, dy$$

$\therefore$ The p.d.f of Y is $g(y) = G'(y)$

$$g(y) = \begin{cases} \dfrac{1}{\Gamma\frac{r}{2}\, 2^{r/2}}\, y^{r/2 - 1}\, e^{-y/2}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

which is the p.d.f of chi-square of $r (\chi^2(r))$

ie) $Y = \frac{2X}{\beta} \sim \chi^2(r)$

Problem :

If X has a gamma distribution with $\alpha = 3$ and $\beta = 4$. find $P_r(3.26 < x < 25.2)$

Soln :

Given $\alpha = 3$ and $\beta = 4$

$\therefore \alpha = \frac{r}{2} \Rightarrow r = 6$

$P_r(3.28 < X < 25.2) = P_r\left(\frac{3.28}{2} < \frac{X}{2} < \frac{25.2}{2}\right)$

$= P_r(1.64 < Y < 12.6)$

Since $Y = \frac{2X}{4} \Rightarrow Y = \frac{X}{2} \sim \chi^2(6)$ $\quad \left[\because Y = \frac{2X}{4},\ Y = \frac{X}{2}\right]$

$P_r(3.28 < X < 25.2) = P_r(Y < 12.6) - P_r(Y < 1.64)$

$= 0.950 - 0.050$

$= 0.900$

$P_r(3.25 < X < 25.2) = 0.900$

Problem :

If $(1-2t)^{-6}$, $t < \frac{1}{2}$ is the m.g.f of the random variable X, find $P_r(X < 5.23)$

Soln:

$$M(t) = (1-2t)^{-6}, \quad t < \frac{1}{2}$$

Here $r = 12$, $\beta = 2$

$\therefore X \sim \chi^2(12)$

$$P_r(X < 5.23) = 0.05$$

Problem :

Let X have a Gamma distribution with p.d.f

$$f(x) = \begin{cases} \frac{1}{\beta^2} x e^{-\frac{x}{\beta}}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

If $x = 2$ is the uniquemode of the distribution find the parameter β and $P_r(X < 9.49)$

Soln:

$$f(x) = \frac{1}{\beta^2} x e^{-\frac{x}{\beta}}$$

$$f'(x) = \frac{1}{\beta^2}\left[x e^{-x/\beta}\left(\frac{-1}{\beta}\right) + e^{-\frac{x}{\beta}}\right]$$

$$f'(x) = \frac{1}{\beta^2} e^{-x/\beta}\left(1 - \frac{x}{\beta}\right)$$

$$f'(x) = 0 \Rightarrow 1 - \frac{x}{\beta} = 0 \Rightarrow \frac{x}{\beta} = 1$$

$$\Rightarrow x = \beta$$

$$f''(x) = \frac{1}{\beta^2}\left[e^{-x/\beta}\left(\frac{-1}{\beta}\right) + \left(1-\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\left(\frac{-1}{\beta}\right)\right]$$

$$= \frac{1}{\beta^3} e^{-x/\beta}\left(1 - \frac{x}{\beta} + 1\right)$$

$$= -\frac{1}{\beta^3} e^{-x/\beta}\left(2 - \frac{x}{\beta}\right)$$

$$f''(x) = \frac{-2}{\beta^3} e^{-\frac{x}{\beta}} + \frac{x}{\beta^4} e^{-x/\beta}$$

$f''(x)$ at $x = \beta \Rightarrow f''(\beta) = \frac{-2}{\beta^3} e^{-1} + \frac{1}{\beta^3} e^{-1}$

$$f''(\beta) = \frac{-1}{\beta^3} e^{-1} < 0$$

$\therefore f(x)$ is maximum at $x = \beta$

$\therefore$ mode of the distribution

$\therefore x = \beta$ is the mode of the distribution

given $x = 2$ is the mode of the distribution

$\therefore \beta = 2$

If X has the gamma distribution with $\alpha = \frac{r}{2}$ then $\frac{2X}{\beta}$ has χ^2 - distribution with r degrees of freedom.

In the given $f(x) = \alpha = 2$ but $\alpha = \frac{r}{2}$

$\therefore r = 4$

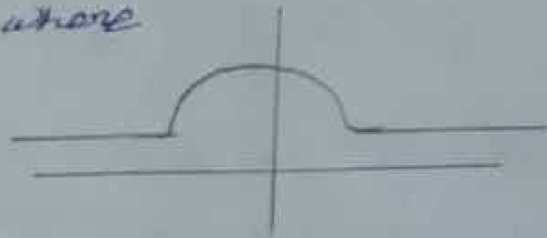
Now, $Y = \frac{2X}{\beta} = \frac{2X}{2} = X$

$\therefore X$ is $\chi^2(4)$ with $r = 4$

From the table $Pr(X < 7.79) = 0.90$

Normal distribution

Let X be a continuous random variable having the p.d.f. $f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}, & -\infty < x < \infty \\ 0, & \text{elsewhere} \end{cases}$



Then X is said to have a normal distribution

Note:

$f(x)$ is a p.d.f

proof:

i) clearly $f(x) \geq 0$

ii) T.P : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\, b} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\, b} \int_{-\infty}^{\infty} e^{-\frac{1}{2} y^2}\, b\,dy \quad , \quad \text{put } y = \frac{x-a}{b}, \quad dy = \frac{1}{b}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\, b} \int_{-\infty}^{\infty} e^{-y^2/2}\, b\,dy$$

Let $I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$

$I^2 = I \cdot I$

$$I^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$$

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx\, dy$$

change in polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$

$\therefore dx\,dy = |J|\, dr\, d\theta$ and $|J| = r$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$

$$|J| = r$$

Also $0 \le r \le \infty$, $0 \le \theta \le 2\pi$

$\therefore dx\,dy = r\,dr\,d\theta$

$$I^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta$$

$$I^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta$$

put $z = \frac{r^2}{2} \Rightarrow dz = r\,dr$

$$\therefore I^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-z}\, dz\, d\theta$$

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi}\left[-e^{-z}\right]_0^{\infty} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = \frac{1}{2\pi}\left[\theta\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}(2\pi) = 1$$

$$I^2 = 1$$

$$I = 1$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = 1$$

$\therefore f(x)$ is a p.d.f

Moment generating function of normal distribution:

$$M(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} e^{-\frac{1}{2b^2}\left((x-a)^2 - 2b^2tx\right)} dx$$

$$M(t) = \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} e^{-\frac{1}{2b^2}\left(x^2 + a^2 - 2x(a+b^2t)\right)}\, dx$$

Now,

$$\left[x - (a+b^2t)\right]^2 = x^2 + (a+b^2t)^2 - 2x(a+b^2t)$$
$$= x^2 + a^2 + b^4t^2 + 2ab^2t - 2x(a+b^2t)$$

$$\left[x - (a+b^2t)\right]^2 - b^4t^2 - 2ab^2t = x^2 + a^2 - 2x(a+b^2t)$$

$$M(t) = \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} e^{-\frac{1}{2b^2}\left[(x-(a+b^2t))^2 - b^4t^2 - 2ab^2t\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} e^{-\frac{1}{2b^2}(x-(a+b^2t))^2} \cdot e^{\frac{1}{2b^2}(b^4t^2 + 2ab^2t)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-(a+b^2t)}{b}\right)^2} \cdot \left(e^{\frac{t^2b^2}{2} + at}\right) dx$$

$$= \frac{e^{at + \frac{t^2b^2}{2}}}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-(a+b^2t)}{b}\right)^2} dx$$

$$= e^{\left(at + \frac{t^2b^2}{2}\right)} \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-(a+b^2t)}{b}\right)^2} dx$$

$$= e^{at + \frac{t^2b^2}{2}} \int_{-\infty}^{\infty} f(x)\, dx$$

$$= e^{at + \frac{b^2t^2}{2}} \cdot 1$$

$$= e^{at + \frac{b^2t^2}{2}}$$

$$M(t) = e^{at + \frac{b^2t^2}{2}}$$

Mean and variance of normal distribution

$$M(t) = e^{at + \frac{b^2t^2}{2}}$$

$$\mu = M'(0)$$

$$M'(t) = e^{at + \frac{b^2t^2}{2}} \times \left(a + \frac{2b^2t}{2}\right)$$

$$M'(t) = e^{at + \frac{b^2t^2}{2}} (a + b^2t)$$

$$\mu = M'(0) = a$$

$$M''(t) = e^{at + \frac{b^2t^2}{2}} \cdot b^2 + (a + b^2t)\, e^{at + \frac{b^2t^2}{2}} (a + b^2t)$$

$$M''(0) = b^2 + a^2$$

$$\sigma^2 = M''(0) - (M'(0))^2$$

$$= b^2 + a^2 - a^2$$

$$\sigma^2 = a^2$$

Note:

i) The p.d.f of normal distribution can be written as $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ -\infty < x < \infty$

ii) The normal distribution is denoted by $n(\mu, \sigma^2)$

iii) The m.g.f of normal distribution can be written as $M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$N(1/2, 1/16)$ Find μ & σ^2

$Y = X_1 + X_2 + X_3 + X_4$ Mean $\mu = a$ & variance $\sigma^2 = b^2$

Standard normal distribution

The normal distribution in which $\mu = 0$ and variance $\sigma^2 = 1$ this known as standard normal distribution and is denoted by $n(0,1)$

The p.d.f of Standard normal distribution is given by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \ -\infty < x < \infty$

The M.g.f of Standard normal distribution is $e^{t^2/2}$

Problem:

Given that x is $n(5,4)$. Find the normal p.d.f

Soln:

The p.d.f of normal distribution $f(x)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ -\infty < x < \infty$$

Here $\mu = 5$ and $\sigma^2 = 4 \Rightarrow \sigma = 2$

$$\therefore f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-5}{2}\right)^2}, \ -\infty < x < \infty$$

Problem:

Given that x has a M.g.f. $M(t) = e^{2t + 32t^2}$

Find μ and σ^2

Soln:

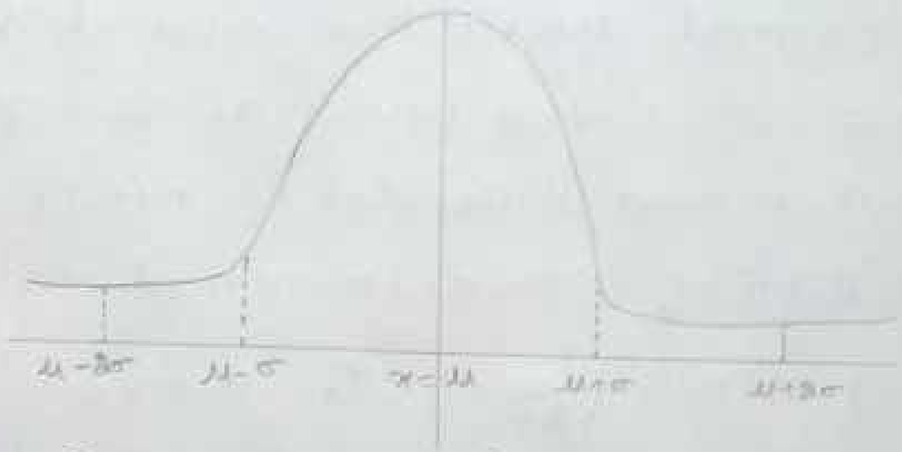
Given $M(t) = e^{2t + 32t^2}$

We have $M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$\therefore \mu = 2$

$\frac{\sigma^2}{2} = 32 \Rightarrow \sigma^2 = 64.$

Properties of normal distribution



1) The Curve of normal distribution is the Bell - shaped one.

2) The normal curve is symmetric about the vertical axis through $x = \mu$.

3) The normal curve attains a maximum value $\frac{1}{\sqrt{2\pi}\,\sigma}$ at $x = \mu$

4) The normal distribution Mean = Median = Mode.

5) For the normal distribution measure of skewness is zero.

6) The curve has its point of inflection at $x = \mu \pm \sigma$ and it is concave downwards if $\mu - \sigma < x < \mu + \sigma$ and concave upwards otherwise.

7) The total area under the curve about the horizontal axis is one.

Theorem : 1

If the random variable X is $n(\mu, \sigma^2)$, $\sigma^2 > 0$ then the random variable $W = \frac{x - \mu}{\sigma}$ is $n(0,1)$

Proof:

Given x is $n(\mu, \sigma^2)$

$\therefore$ The p.d.f of x is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$

Now, the distribution function $G(w)$ of W

$$G(w) = P_r(W \le w)$$
$$= P_r\left(\frac{X-\mu}{\sigma} \le w\right)$$
$$= P_r(x \le \mu + \sigma w)$$
$$= \int_{-\infty}^{\mu+\sigma w} f(x)\,dx$$
$$= \int_{-\infty}^{\mu+\sigma w} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $w = \frac{x-\mu}{\sigma} \Rightarrow dx = \sigma\,dw$

When $x = -\infty \Rightarrow w = -\infty$

When $x = \mu + \sigma w \Rightarrow w = w$

$$G(w) = \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}w^2} \sigma\,dw$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{w^2}{2}} dw$$

$\therefore g(w) = G'(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}, \quad -\infty < w < \infty$

Which is the p.d.f of standard normal distribution

$\therefore W$ is $n(0,1)$

Note:

Let x be $n(0,1)$, $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$.

Result : 1

$N(-x) = 1 - N(x)$

Proof:

$$N(-x) = \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$$

Also $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw = 1$

ie) $\displaystyle\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw + \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw = 1$

$$\int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw = 1 - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$$

$$\int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw = 1 - N(x)$$

Let $y = -w \Rightarrow dy = -dw$

When $w = x \Rightarrow y = -x$

When $w = \infty \Rightarrow y = -\infty$

$$\int_{-x}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} (-dy) = 1 - N(x)$$

$$\int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = 1 - N(x)$$

$$N(-x) = 1 - N(x)$$

Result : 2

If x is $n(\mu, \sigma^2)$ then $P_r(X < c) = N\left(\frac{c-\mu}{\sigma}\right)$

Proof:

If x is $n(\mu, \sigma^2)$ then $W = \frac{X-\mu}{\sigma}$ is $n(0,1)$

$$P_r(X < C) = P_r\left(\frac{X-\mu}{\sigma} < \frac{C-\mu}{\sigma}\right)$$

$$= P_r\left(W < \frac{c-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{\frac{c-\mu}{\sigma}} f(w)\,dw = \int_{-\infty}^{\frac{c-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\,dw$$

$$= N\left(\frac{c-\mu}{\sigma}\right)$$

Result: 3

If x is $n(\mu, \sigma^2)$, then $P_r(c_1 < x < c_2) = N\left(\frac{c_2-\mu}{\sigma}\right) - N\left(\frac{c_1-\mu}{\sigma}\right)$

Proof:

If x is $n(\mu, \sigma^2)$ then $W = \frac{x-\mu}{\sigma}$ is $n(0,1)$

Now, $P_r(c_1 < x < c_2) = P_r(x < c_2) - P_r(x < c_1)$

$$= N\left(\frac{c_2-\mu}{\sigma}\right) - N\left(\frac{c_1-\mu}{\sigma}\right)$$

Problem:

Let x be $n(2, 25)$ i) $P_r(x < 10)$ ii) $P_r(0 < x < 10)$ iii) $P_r(-8 < x < 1)$

Soln:

Here $\mu = 2$, $\sigma^2 = 25 \Rightarrow \sigma = 5$

i) $P_r(x < 10) = N\left(\frac{10-2}{5}\right) = N(1.6) = 0.945$

ii) $P_r(0 < x < 10) = N\left(\frac{10-2}{5}\right) - N\left(\frac{0-2}{5}\right)$

$= 0.945 - N(-0.4)$

$= 0.945 - (1 - N(0.4))$

$= 0.945 - (1 - 0.655)$

$= 0.945 - 1 + 0.655$

$= 0.6$

iii) $P_r(-8 < x < 1) = N\left(\frac{1-2}{5}\right) - N\left(\frac{-8-2}{5}\right)$

$= N(-1/5) - N(-2)$

$= (1 - N(0.2)) - (1 - N(2))$

$= (1 - 0.579) - (1 - 0.977)$

$= 0.398.$

Problem:

Let x be $N(\mu, \sigma^2)$ Show that

$P_r(\mu - 2\sigma < x < \mu + 2\sigma) = 0.954$

Soln:

$P_r(\mu - 2\sigma < x < \mu + 2\sigma) = P_r(x < \mu + 2\sigma) - P_r(x < \mu - 2\sigma)$

$= N\left(\frac{\mu + 2\sigma - \mu}{\sigma}\right) - N\left(\frac{\mu - 2\sigma - \mu}{\sigma}\right)$

$= N(2) - N(-2)$

$= N(2) - (1 - N(2)) = N(2) - 1 + N(2)$

$= 2N(2) - 1$

$= 2(0.977) - 1$

$= 0.954$

Problem:

If x is $n(75, 100)$ find $P_r(x < 60)$ and $P_r(70 < x < 100)$

Let x be $n(\mu, \sigma^2)$, so that $P_r(x < 89) = 0.90$,

$P_r(x < 94) = 0.95$. Find μ, σ^2.

Soln: Given x is $n(75, 100)$

Here $\mu = 75$, $\sigma^2 = 100 \Rightarrow \sigma = 10$

$P_r(x < 60) = N\left(\frac{60-75}{10}\right) = N\left(\frac{-15}{10}\right)$

$= N(-1.5) = 1 - N(1.5)$

$= 1 - 0.933$

$= 0.067$

$$P_r(70 < X < 100) = N\left(\frac{100-75}{10}\right) - N\left(\frac{70-75}{10}\right)$$

$= N(2.5) - N(-0.5)$

$= N(2.5) - [1 - N(0.5)]$

$= 0.994 - [1 - 0.691]$

$= 0.685.$

Let x be $n(\mu, \sigma^2)$

$P_r(X < 89) = 0.90$

$N\left(\frac{89-\mu}{\sigma}\right) = 0.90 = N(1.282)$

$\frac{89-\mu}{\sigma} = 1.282$

$89 = 1.282\sigma + \mu \rightarrow ①$

$P_r(x < 94) = 0.95$

$N\left(\frac{94-\mu}{\sigma}\right) = 0.95 = N(1.645)$

$\frac{94-\mu}{\sigma} = 1.645$

$94 = 1.645\sigma + \mu \rightarrow ②$

solving ① & ②

$1.282\sigma - 1.645\sigma = 5$

$0.363\sigma = 5$

$\sigma = \frac{5}{0.363} = \frac{5000}{363}$

$\boxed{\sigma = 13.774}$

$① \Rightarrow 1.282(13.774) + \mu = 89$

$\boxed{\mu = 71.342}$

Example:

Suppose that 10% of the probability for a certain distribution. (i.e) $n(\mu, \sigma^2)$. is below 60 and that 5% above 90 what are all the value of μ & σ

Soln:

Given that $P(X < 60) = 0.10$

$P(X > 90) = 0.05$

$1 - P(X < 90) = 0.05$

$P(X < 90) = 0.95$; $N\left(\frac{90-\mu}{\sigma}\right) = 0.95$

$N\left(\frac{60-\mu}{\sigma}\right) = 0.10$

$= 0 - 0.90 \Rightarrow N(-x)$

$N\left(\frac{60-\mu}{\sigma}\right) = N(-1.282)$

$\frac{60-\mu}{\sigma} = -1.282$

$\frac{60-\mu}{\sigma} = -1.282$

$60 - \mu = -1.282\,\sigma$

$-1.282\sigma + \mu = 60$ —— ①

$N\left(\frac{90-\mu}{\sigma}\right) = 0.95 \Rightarrow N(1.645)$

$\frac{90-\mu}{\sigma} = 1.645 \Rightarrow 1.645\sigma + \mu = 90$

② − ① $\Rightarrow$ $2.927\sigma = 30$

$\boxed{\sigma = 10.249}$

① $\Rightarrow$ $-1.282(10.249) + \mu = 60$

$\boxed{\mu = 73.14}$

Theorem:

If the random variable X is $n(\mu, \sigma^2)$ where $\sigma^2 > 0$, then the random variable $Y = \left(\frac{X-\mu}{\sigma}\right)^2$ is $\chi^2(1)$

Proof:

Given X is $n(\mu, \sigma^2)$

W.K.T. If the random variable X is $n(\mu, \sigma^2)$; $\sigma^2 > 0$ then the random variable $W = \frac{X-\mu}{\sigma}$ is $n(0,1)$.

Let $Y = \left(\frac{X-\mu}{\sigma}\right)^2$

(ie) $Y = W^2$.

The distribution function of Y is $G(Y) = P_r(Y \leq y)$

$$\Rightarrow P_r(W^2 \leq y)$$

$$= P_r(|W| \leq \sqrt{y})$$

$$= P_r(-\sqrt{y} < W < \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2\pi} e^{-w^2/2} \, dw$$

$$= 2 \int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \, dw \quad \text{——— (1)}$$

Put $t = w^2$; $\Rightarrow w = \sqrt{t}$

$dt = 2w\,dw \Rightarrow 2\sqrt{t}\,dw$

$dw = \frac{dt}{2\sqrt{t}}$

w	0	$\sqrt{y}$
t	0	y

$$(1) \Rightarrow G(y) = 2 \int_{0}^{y} \frac{1}{\sqrt{2\pi}} e^{-t/2} \frac{1}{2\sqrt{t}} \, dt$$

$$= \int_0^9 \frac{1}{2^{1/2}\sqrt{1/2}} e^{-1/2} t^{1/2-1}$$

The p.d.f of y is $G'(y) = g(y)$

$g(y) = G_1'(y) = \frac{1}{2^{1/2}\sqrt{1/2}} e^{-y/2} . y^{1/2-1}$ which is the p.d.f of χ^2 – distribution with $r = 1$.

Problem:

If x is $n(1, 4)$, find $P_r(1 < x^2 < 9)$.

Soln:

Here $\mu = 1$; $\sigma^2 = 4 \Rightarrow \sigma = 2$

$P_r(1 < x^2 < 9) = P_r(\sqrt{1} < |x| < \sqrt{9})$

$= P_r(1 < |x| < 3)$

$= P_r(-3 < X < -1) + P(1 < X < 3)$

$= P_r(X < -1) - P_r(X < -3) + P_r(X < 3) - P_r(X < 1)$

$= N\left(\frac{-1-1}{2}\right) - N\left(\frac{-3-1}{2}\right) + N\left(\frac{3-1}{2}\right) - N\left(\frac{1-1}{2}\right)$

$= N(-1) - N(-2) + N(1) - N(0)$

$= 1 - N(1) - (1 - N(2)) + N(1) - N(0)$

$= N(2) - N(0)$

$= 0.977 - 0.500$

$= 0.477$

Problem

If e^{3t+8t^2} is the m.g.f of the random variable X.

Find $P_r(-1 < X < 9)$

Solution:

Here $M(t) = e^{+\mu t + \frac{\sigma^2 t^2}{2}}$

Given $M(t) = e^{3t + 8t^2}$

Here $\mu = 3$; $\frac{\sigma^2}{2} = 8 \Rightarrow \sigma^2 = 16 \Rightarrow \sigma = 4$

$P_r(-1 < X < 9) = P_r(X < 9) - P_r(X < -1)$

$= N\left(\frac{9-3}{4}\right) - N\left(\frac{-1-3}{4}\right)$

$= N(6/4) - N(-1)$

$= N(1.5) - (1 - N(1))$

$= 0.933 - 1 + 0.841$

$= 0.774$

Problem:

If X is $n(\mu, \sigma^2)$ then $E(|X-\mu|) = \sigma\sqrt{\frac{2}{\pi}}$

Soln:

Given X is a $n(\mu, \sigma^2)$;

The pdf of X is $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Now,

$$E(|X-\mu|) = \int_{-\infty}^{\infty} |x-\mu| f(x)\,dx$$

$$= \int_{-\infty}^{\infty} |x-\mu| \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $y = x - \mu \Rightarrow dy = dx$

$$E(|Y|) = \int_{-\infty}^{\infty} |y| \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2} y^2/\sigma^2} dy$$

$$= \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}} \int_0^\infty e^{-w}\, dw$$

$$= \sigma \sqrt{\frac{2}{\pi}} \left[\frac{e^{-w}}{-1} \right]_0^\infty$$

$$= \sigma \sqrt{\frac{2}{\pi}} \left[-(0-1) \right] \Rightarrow \sigma \sqrt{\frac{2}{\pi}}$$

(ie) $E(|X-\mu|) = \sigma \sqrt{\frac{2}{\pi}}$

Problem:

If X is $n(\mu, \sigma^2)$, find b so that $P_r\left(-b < \frac{x-\mu}{\sigma} < b\right) = 0.90$

Soln:

If X is $n(\mu, \sigma^2)$, then $W = \frac{x-\mu}{\sigma}$ is $n(0,1)$

Now, $P_r\left(-b < \frac{X-\mu}{\sigma} < b\right) = 0.90$

$$P_r(-\sigma b < X-\mu < b\sigma) = 0.90$$

$$P_r(\mu - \sigma b < X < \mu + b\sigma) = 0.90$$

$$N\left(\frac{\mu + b\sigma - \mu}{\sigma}\right) - N\left(\frac{\mu - b\sigma - \mu}{\sigma}\right) = 0.90$$

$$N(b) - N(-b) = 0.90$$

$$N(b) - (1 - N(b)) = 0.90$$

$$2(N(b) - 1) = 0.90$$

$$2N(b) = 1.90$$

$$N(b) = 0.95$$

$$b = 1.645$$